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1998 J. Phys. A: Math. Gen. 31 1405

(http://iopscience.iop.org/0305-4470/31/5/010)

J. Phys. A: Math. Gen. **31** (1998) 1405–1414. Printed in the UK PII: S0305-4470(98)84408-9

Bäcklund transformations and nonlinear superposition formulae of a differential-difference KdV equation

Xing-Bao Hu†‡§ and Peter A Clarkson‡‖

† State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China

‡ Institute of Mathematics and Statistics, University of Kent at Canterbury, Canterbury, Kent CT2 7NF, UK

Received 22 May 1997

Abstract. In this paper, a differential-difference KdV equation is considered. We present two Bäcklund transformations and corresponding nonlinear superposition formulae for it. As an application of the obtained results, N-soliton solutions first obtained by Ohta and Hirota are rederived. Furthermore, a sequence of rational solutions are also obtained. Thus the integrability of the differential-difference KdV equation is further confirmed.

1. Introduction

Recently, Ohta and Hirota [1] proposed a differential-difference KdV equation

$$\frac{4}{1+a^2un}\frac{\mathrm{d}u(n)}{\mathrm{d}t} = \Delta^3 Mu(n) + 6u(n)\Delta Mu(n) + a^2[\Delta M(u(n)\Delta^2 u(n)) +3(\Delta Mu(n))(\Delta^2 u(n))] \tag{1}$$

where Δ and M are the difference and averaging operators defined by

$$\Delta F(n) = \tfrac{1}{a}[F(n+\tfrac{1}{2}a) - F(n-\tfrac{1}{2}a)] \qquad MF(n) = \tfrac{1}{2}[F(n+\tfrac{1}{2}a) + F(n-\tfrac{1}{2}a)].$$

In the continuum limit as $a \longrightarrow 0$, Δ and M become $\partial/\partial x$ and 1, respectively, and equation (1) reduces to the KdV equation

$$4u_t = u_{xxx} + 6uu_x. \tag{2}$$

Through the variable transformation

$$u(n) = \frac{1}{a^2}\left\{\frac{f(n+2a)f(n-a)}{f(n+a)f(n)} - 1\right\}$$

equation (1) is transformed into the bilinear form

$$(2a\mathrm{D}_z + 1)f(n+a)\cdot f(n) = f(n+2a)f(n-a) \tag{3}$$

$$(8a^3\mathrm{D}_t - 3)f(n+a)\cdot f(n) = (2a\mathrm{D}_z - 3)f(n+2a)\cdot f(n-a) \tag{4}$$

§ E-mail address: x.b.hu@ukc.ac.uk

‖ E-mail address: p.a.clarkson@ukc.ac.uk

or

$$[2a\mathrm{D}_z \sinh(\tfrac{1}{2}a\mathrm{D}_n) + \cosh(\tfrac{1}{2}a\mathrm{D}_n) - \cosh(\tfrac{3}{2}a\mathrm{D}_n)] f(n)\cdot f(n) = 0 \tag{5}$$

$$[8a^3\mathrm{D}_t \sinh(\tfrac{1}{2}a\mathrm{D}_n) - 3\cosh(\tfrac{1}{2}a\mathrm{D}_n) - 2a\mathrm{D}_z \sinh(\tfrac{3}{2}a\mathrm{D}_n) + 3\cosh(\tfrac{3}{2}a\mathrm{D}_n)] f(n)\cdot f(n) = 0 \tag{6}$$

where z is an auxiliary variable and the bilinear operators are defined as follows [2–4]:

$$\mathrm{D}_x^m \mathrm{D}_t^n b\cdot c \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n b(x,t)c(x',t')|_{x'=x,t'=t}$$

$$\exp(\delta\,\mathrm{D}_n) b(n)\cdot c(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] b(n)c(n')|_{n'=n} = b(n+\delta)c(n-\delta).$$

On the other hand, it is known that the KdV equation (2) also has the differential-difference analogue [3, 5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n}{1+w_n}\right) = w_{n-1/2} - w_{n+1/2}. \tag{7}$$

However, just as pointed out by Ohta and Hirota [1], equation (7) does not converge to the KdV equation in the continuum limit and it is necessary when deriving the KdV equation to use the coordinates which move with an infinite velocity.

It has been shown in [1] that equations (5) and (6) possess N-soliton solutions. Therefore, equations (5) and (6), or equivalently (1), are thought to be integrable in the sense of passing the so-called N-soliton test. In this paper, we consider the differential-difference KdV equations (5) and (6) and present two different Bäcklund transformations and corresponding nonlinear superposition formulae. As an application of the obtained results, N-soliton solutions first obtained by Ohta and Hirota are rederived. Furthermore, a sequence of rational solutions are also obtained. Thus the integrability of (5) and (6) or (1) is confirmed.

This paper is organized as follows. In section 2, two different Bäcklund transformations for the differential-difference KdV equations (5) and (6) are given. We present the corresponding nonlinear superposition formulae of (5) and (6) in sections 3 and 4, respectively. N-soliton solutions and a sequence of rational solutions are obtained. In section 5, a conclusion and discussion are given. Finally, we list some bilinear operator identities which are used in this paper in the appendix.

2. Bäcklund transformations for the differential-difference KdV equation

In this section we give two different Bäcklund transformations for equations (5) and (6). These are given in the following propositions.

Proposition 1. A Bäcklund transformation for (5) and (6) is

$$\exp(-\tfrac{1}{2}a\mathrm{D}_n) f(n)\cdot f'(n) = [\lambda\exp(\tfrac{3}{2}a\mathrm{D}_n) + \mu\exp(\tfrac{1}{2}a\mathrm{D}_n)] f(n)\cdot f'(n) \tag{8}$$

$$[2a\mathrm{D}_z + \lambda\exp(2a\mathrm{D}_n) + \gamma] f(n)\cdot f'(n) = 0 \tag{9}$$

$$[8a^3\mathrm{D}_t + 2\lambda a\mathrm{D}_z\exp(2a\mathrm{D}_n) - 2\lambda\exp(2a\mathrm{D}_n) + \lambda\gamma\exp(2a\mathrm{D}_n) + k] f(n)\cdot f'(n) = 0 \tag{10}$$

where λ, μ, γ and k are arbitrary constants.

Proposition 2. A Bäcklund transformation for (5) and (6) is

$$[\cosh(a\mathrm{D}_n) - \lambda] f(n) \cdot f'(n) = 0 \tag{11}$$

$$[a\mathrm{D}_z - \lambda \sinh(a\mathrm{D}_n)] f(n) \cdot f'(n) = 0 \tag{12}$$

$$[2a^3\mathrm{D}_t + 2\lambda \sinh(a\mathrm{D}_n) - a\lambda^2\mathrm{D}_z - a\lambda\mathrm{D}_z \cosh(a\mathrm{D}_n) + \mu] f(n) \cdot f'(n) = 0 \tag{13}$$

where λ and μ are arbitrary constants.

In the following we only give the details of the proof of proposition 2. Proposition 1 can be proved in a similar way.

Proof of proposition 2. Let $f(n)$ be a solution of (5) and (6). If we can find three equations which relate $f(n)$ and $f'(n)$ such that

$$P_1 \equiv [2a\mathrm{D}_z \sinh(\tfrac{1}{2}a\mathrm{D}_n) + \cosh(\tfrac{1}{2}a\mathrm{D}_n) - \cosh(\tfrac{3}{2}a\mathrm{D}_n)] f'(n) \cdot f'(n) = 0$$

$$P_2 \equiv [8a^3\mathrm{D}_t \sinh(\tfrac{1}{2}a\mathrm{D}_n) - 3\cosh(\tfrac{1}{2}a\mathrm{D}_n) - 2a\mathrm{D}_z \sinh(\tfrac{3}{2}a\mathrm{D}_n) + 3\cosh(\tfrac{3}{2}a\mathrm{D}_n)] f'(n) \cdot f'(n) = 0$$

then this is a Bäcklund transformation. Here we show that (11)–(13) indeed provide a Bäcklund transformation for (5) and (6).

Analogous to the proof in [5], we know that $P_1 = 0$ can be proved using (11) and (12). Thus it suffices to show that $P_2 = 0$. Making use of (A1), (A2) and (A4), we have

$$\begin{aligned} -[\cosh(\tfrac{1}{2}a\mathrm{D}_n) f(n) \cdot f(n)] P_2 &= 16a^3 \sinh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_t f(n) \cdot f'(n)) \cdot f(n) f'(n) \\ &\quad + 12 \sinh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad - 4a\mathrm{D}_z \cosh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad - 4a \sinh(\tfrac{1}{2}a\mathrm{D}_n)\{[\mathrm{D}_z \cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad + [\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\mathrm{D}_z \sinh(a\mathrm{D}_n) f(n) \cdot f'(n)]\}. \end{aligned} \tag{14}$$

On the other hand, using the fact that $P_1 = 0$ and $f(n)$ is a solution of (5) we know that

$$\begin{aligned} &[\cosh(\tfrac{3}{2}a\mathrm{D}_n) f'(n) \cdot f'(n)][2a\mathrm{D}_z \sinh(\tfrac{1}{2}a\mathrm{D}_n) + \cosh(\tfrac{1}{2}a\mathrm{D}_n) - \cosh(\tfrac{3}{2}a\mathrm{D}_n)] f(n) \cdot f(n) \\ &\quad - [\cosh(\tfrac{3}{2}a\mathrm{D}_n) f(n) \cdot f(n)][2a\mathrm{D}_z \sinh(\tfrac{1}{2}a\mathrm{D}_n) + \cosh(\tfrac{1}{2}a\mathrm{D}_n) \\ &\quad - \cosh(\tfrac{3}{2}a\mathrm{D}_n)] f'(n) \cdot f'(n) = 0 \end{aligned}$$

from which it follows, by using (A2) and (A5), that

$$\begin{aligned} &a \sinh(\tfrac{1}{2}a\mathrm{D}_n)\{[\mathrm{D}_z \cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad + [\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\mathrm{D}_z \sinh(a\mathrm{D}_n) f(n) \cdot f'(n)]\} \\ &= a\mathrm{D}_z \cosh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad - \sinh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)]. \end{aligned} \tag{15}$$

Thus (14) becomes, by using (11), (15), (A3) and (A6),

$$\begin{aligned} -[\cosh(\tfrac{1}{2}a\mathrm{D}_n) f(n) \cdot f(n)] P_2 &= 16a^3 \sinh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_t f(n) \cdot f'(n)) \cdot f(n) f'(n) \\ &\quad + 16 \sinh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &\quad - 8a\mathrm{D}_z \cosh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \\ &= 16a^3 \sinh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_t f(n) \cdot f'(n)) \cdot f(n) f'(n) \\ &\quad + 16 \sinh(\tfrac{1}{2}a\mathrm{D}_n)[\lambda \sinh(a\mathrm{D}_n) f(n) \cdot f'(n)] \cdot f(n) f'(n) \\ &\quad - 8a \sinh(\tfrac{1}{2}a\mathrm{D}_n)\{(\mathrm{D}_z f(n) \cdot f'(n)) \cdot [\cosh(a\mathrm{D}_n) f(n) \cdot f'(n)] \end{aligned}$$

$$
\begin{aligned}
&+[\mathrm{D}_z \cosh(a\mathrm{D}_n) f(n)\cdot f(n)]\cdot f(n)f'(n)\} \\
&= 8\sinh(\tfrac{1}{2}a\mathrm{D}_n)\{[2a^3\mathrm{D}_t + 2\lambda\sinh(a\mathrm{D}_n) - a\lambda^2\mathrm{D}_z \\
&\quad -a\lambda\mathrm{D}_z\cosh(a\mathrm{D}_n)] f(n)\cdot f'(n)\}\cdot f(n)f'(n) \\
&= 0.
\end{aligned}
$$

Thus we have completed the proof of proposition 2. □

3. A nonlinear superposition formula and soliton solutions

In this section, we shall give a nonlinear superposition formula for equations (5) and (6) corresponding to the Bäcklund transformation (8)–(10). The result obtained is as follows.

Proposition 3. Let $f_0(n)$ be a solution of the differential-difference KdV equations (5) and (6) and suppose that $f_i(n)$, $i = 1, 2$, is a solution of (5) and (6), which is related by $f_0(n)$ under the Bäcklund transformation (8)–(10) with $(\lambda_i, \mu_i, \gamma_i, k_i)$, i.e. $f_0(n) \xrightarrow{(\lambda_i,\mu_i,\gamma_i,k_i)} f_i(n)$, for $i = 1, 2$, with $\lambda_1\lambda_2\mu_1\mu_2 \neq 0$ and $f_j(n) \neq 0$, $j = 0, 1, 2$. Then $f_{12}(n)$ defined by

$$
\exp(a\mathrm{D}_n) f_0(n)\cdot f_{12}(n) = c[\lambda_1\exp(-a\mathrm{D}_n) - \lambda_2\exp(a\mathrm{D}_n)] f_1(n)\cdot f_2(n) \tag{16}
$$

where c is an arbitrary non-zero constant, is a new solution of (5) and (6) which is related to $f_1(n)$ and $f_2(n)$ under the Bäcklund transformation (8)–(10) with parameters $(\lambda_2, \mu_2, \gamma_2, k_2)$ and $(\lambda_1, \mu_1, \gamma_1, k_1)$, respectively.

Proof. Analogous to the deduction of [6], we can show that

$$
\exp(\tfrac{3}{2}a\mathrm{D}_n) f_0\cdot f_{12} = c[-\mu_1\exp(-\tfrac{1}{2}a\mathrm{D}_n) + \mu_2\exp(\tfrac{1}{2}a\mathrm{D}_n)] f_1\cdot f_2 \tag{17}
$$

$$
\exp(\tfrac{1}{2}a\mathrm{D}_n) f_0\cdot f_{12} = c[-\mu_1\lambda_2\exp(\tfrac{1}{2}a\mathrm{D}_n) + \lambda_1\mu_2\exp(-\tfrac{1}{2}a\mathrm{D}_n)] f_1\cdot f_2 \tag{18}
$$

$$
\exp(-\tfrac{1}{2}a\mathrm{D}_n) f_1\cdot f_{12} = [\lambda_2\exp(\tfrac{3}{2}a\mathrm{D}_n) + \mu_2\exp(\tfrac{1}{2}a\mathrm{D}_n)] f_1\cdot f_{12} \tag{19}
$$

$$
\exp(-\tfrac{1}{2}a\mathrm{D}_n) f_2\cdot f_{12} = [\lambda_1\exp(\tfrac{3}{2}a\mathrm{D}_n) + \mu_1\exp(\tfrac{1}{2}a\mathrm{D}_n)] f_2\cdot f_{12} \tag{20}
$$

$$
[2a\mathrm{D}_z + \lambda_2\exp(2a\mathrm{D}_n) + \gamma_2] f_1\cdot f_{12} = 0 \tag{21}
$$

$$
[2a\mathrm{D}_z + \lambda_1\exp(2a\mathrm{D}_n) + \gamma_1] f_2\cdot f_{12} = 0. \tag{22}
$$

Thus, in order to prove the proposition, it suffices to show that

$$
[8a^3\mathrm{D}_t + 2\lambda_2 a\mathrm{D}_z\exp(2a\mathrm{D}_n) - 2\lambda_2\exp(2a\mathrm{D}_n) + \lambda_2\gamma_2\exp(2a\mathrm{D}_n) + k_2] f_1\cdot f_{12} = 0 \tag{23}
$$

$$
[8a^3\mathrm{D}_t + 2\lambda_1 a\mathrm{D}_z\exp(2a\mathrm{D}_n) - 2\lambda_1\exp(2a\mathrm{D}_n) + \lambda_1\gamma_1\exp(2a\mathrm{D}_n) + k_1] f_2\cdot f_{12} = 0. \tag{24}
$$

Since $f_1(n)$ and $f_2(n)$ are two solutions of (5) and (6), then, analogous to the proof of proposition 2, we have that

$$
\begin{aligned}
0 &= [\cosh(\tfrac{1}{2}a\mathrm{D}_n) f_2\cdot f_2][8a^3\mathrm{D}_t\sinh(\tfrac{1}{2}a\mathrm{D}_n) - 3\cosh(\tfrac{1}{2}a\mathrm{D}_n) - 2a\mathrm{D}_z\sinh(\tfrac{3}{2}a\mathrm{D}_n) \\
&\quad +3\cosh(\tfrac{3}{2}a\mathrm{D}_n)] f_1\cdot f_1 - [\cosh(\tfrac{1}{2}a\mathrm{D}_n) f_1\cdot f_1][8a^3\mathrm{D}_t\sinh(\tfrac{1}{2}a\mathrm{D}_n) \\
&\quad -3\cosh(\tfrac{1}{2}a\mathrm{D}_n) - 2a\mathrm{D}_z\sinh(\tfrac{3}{2}a\mathrm{D}_n) + 3\cosh(\tfrac{3}{2}a\mathrm{D}_n)] f_2\cdot f_2 \\
&\quad +[\cosh(\tfrac{3}{2}a\mathrm{D}_n) f_2\cdot f_2][2a\mathrm{D}_z\sinh(\tfrac{1}{2}a\mathrm{D}_n) + \cosh(\tfrac{1}{2}a\mathrm{D}_n) \\
&\quad -\cosh(\tfrac{3}{2}a\mathrm{D}_n)] f_1\cdot f_1 - [\cosh(\tfrac{3}{2}a\mathrm{D}_n) f_1\cdot f_1][2a\mathrm{D}_z\sinh(\tfrac{1}{2}a\mathrm{D}_n) \\
&\quad +\cosh(\tfrac{1}{2}a\mathrm{D}_n) - \cosh(\tfrac{3}{2}a\mathrm{D}_n)] f_2\cdot f_2 \\
&= 16a^3\sinh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_t f_1\cdot f_2)\cdot f_1 f_2 \\
&\quad +4\sinh(a\mathrm{D}_n)[\exp(\tfrac{1}{2}a\mathrm{D}_n) f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n) f_1\cdot f_2]
\end{aligned}
$$

$$-4a\mathrm{D}_z\cosh(a\mathrm{D}_n)\cdot[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$

[by (A1), (A2), (A4), (A5)]

$$= 8a^3\mathrm{D}_t[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$
$$+4\sinh(a\mathrm{D}_n)[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$
$$-4a\mathrm{D}_z\cosh(a\mathrm{D}_n)[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$

[by (A7)]

$$= 8a^3\mathrm{D}_t\left[-\frac{1}{c\mu_1\lambda_2}\exp(\tfrac{1}{2}a\mathrm{D}_n)f_0\cdot f_{12}\right]\cdot[\exp(-\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$
$$-4\sinh(a\mathrm{D}_n)\left[-\frac{1}{c\mu_1}\exp(\tfrac{3}{2}a\mathrm{D}_n)f_0\cdot f_{12}\right]\cdot[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$
$$+4a\mathrm{D}_z\cosh(a\mathrm{D}_n)\left[-\frac{1}{c\mu_1}\exp(\tfrac{3}{2}a\mathrm{D}_n)f_0\cdot f_{12}\right]\cdot[\exp(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]$$

[by (17), (18)]

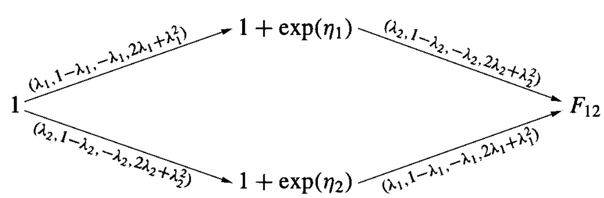
$$= -\frac{8a^3}{c\mu_1\lambda_2}\exp(\tfrac{1}{2}a\mathrm{D}_n)[(\mathrm{D}_t f_0\cdot f_2)\cdot f_1f_{12}-f_0f_2\cdot(\mathrm{D}_t f_1\cdot f_{12})]$$
$$+\frac{2}{c\mu_1}\exp(\tfrac{1}{2}a\mathrm{D}_n)[(\exp(2a\mathrm{D}_n)f_0\cdot f_2)\cdot f_1f_{12}-f_0f_2\cdot(\exp(2a\mathrm{D}_n)f_1\cdot f_{12})]$$
$$-\frac{2a}{c\mu_1}\exp(\tfrac{1}{2}a\mathrm{D}_n)[(\mathrm{D}_z\exp(2a\mathrm{D}_n)f_0\cdot f_2)\cdot f_1f_{12}$$
$$-(\exp(2a\mathrm{D}_n)f_0\cdot f_2)\cdot(\mathrm{D}_z f_1\cdot f_{12})]$$
$$-\frac{2a}{c\mu_1}\exp(\tfrac{1}{2}a\mathrm{D}_n)[(\mathrm{D}_z f_0\cdot f_2)\cdot(\exp(2a\mathrm{D}_n)f_1\cdot f_{12})$$
$$-f_0f_2\cdot(\mathrm{D}_z\exp(2a\mathrm{D}_n)f_1\cdot f_{12})]$$

[by (A8)–(A10)]

$$= -\frac{1}{c\mu_1\lambda_2}\exp(\tfrac{1}{2}a\mathrm{D}_n)\{[8a^3\mathrm{D}_t-2\lambda_2\exp(2a\mathrm{D}_n)+2a\lambda_2\mathrm{D}_z\exp(2a\mathrm{D}_n)$$
$$+\lambda_2\gamma_2\exp(2a\mathrm{D}_n)]f_0\cdot f_2\}\cdot f_1f_{12}$$
$$+\frac{1}{c\mu_1\lambda_2}\exp(\tfrac{1}{2}a\mathrm{D}_n)f_0f_2\cdot\{[8a^3\mathrm{D}_t-2\lambda_2\exp(2a\mathrm{D}_n)+2a\lambda_2\mathrm{D}_z\exp(2a\mathrm{D}_n)$$
$$+\lambda_2\gamma_2\exp(2a\mathrm{D}_n)]f_1\cdot f_{12}\}$$
$$= \frac{1}{c\mu_1\lambda_2}\exp(\tfrac{1}{2}a\mathrm{D}_n)f_0f_2\cdot\{[8a^3\mathrm{D}_t-2\lambda_2\exp(2a\mathrm{D}_n)+2a\lambda_2\mathrm{D}_z\exp(2a\mathrm{D}_n)$$
$$+\lambda_2\gamma_2\exp(2a\mathrm{D}_n)+k_2]f_1\cdot f_{12}\}$$

which implies that (23) holds. Similarly we can show that (24) holds. Thus we have completed the proof of proposition 3. □

As an application of this result, we give some soliton solutions. Choose $f_0(n)=1$, $c=1/(\lambda_1-\lambda_2)$. It is easily verified that



where

$$F_{12} = 1 + \frac{\lambda_1\lambda_2 - 1}{\lambda_1 - \lambda_2}\exp(\eta_1) + \frac{1 - \lambda_1\lambda_2}{\lambda_1 - \lambda_2}\exp(\eta_2) - \exp(\eta_1 + \eta_2)$$

with, for $i = 1, 2$,

$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0 \qquad \lambda_i = -\exp(ap_i)$$
$$q_i = \frac{1}{a}\sinh(ap_i) \qquad r_i = \frac{1}{4a^3}\sinh(2ap_i) - \frac{1}{2a^3}\sinh(ap_i).$$

Thus F is a two-soliton solution of (5) and (6). In general, continuing along this line, we can easily re-derive multi-soliton solutions of the differential-difference KdV equation first obtained by Ohta and Hirota.

4. Nonlinear superposition formula and rational solutions

In this section, we turn to consider the Bäcklund transformation (11)–(13). In the following, we just focus on special Bäcklund parameters, i.e. $\lambda = 1$, $\mu = 0$. In this case, (11)–(13) become

$$[\cosh(a\mathrm{D}_n) - 1]f(n)\cdot f'(n) = 0 \tag{25}$$
$$[a\mathrm{D}_z - \sinh(a\mathrm{D}_n)]f(n)\cdot f'(n) = 0 \tag{26}$$
$$[2a^3\mathrm{D}_t + a\mathrm{D}_z - a\mathrm{D}_z\cosh(a\mathrm{D}_n)]f(n)\cdot f'(n) = 0. \tag{27}$$

We shall represent the transformation (25)–(27) symbolically by $f(n) \to f'(n)$. Now let $f_0(n)$ and $f_1(n)$ be two solutions of (5) and (6) and $f_0(n) \to f_1(n)$, with $f_0(n) \neq 0$ and $f_1(n) \neq 0$. Suppose that $f_2(n)$ is given by

$$\sinh(\tfrac{1}{2}a\mathrm{D}_n)f_1(n)\cdot f_2(n) = k\cosh(\tfrac{1}{2}a\mathrm{D}_n)f_0(n)\cdot f_0(n) \tag{28}$$

where k is a non-zero constant. From these assumptions and analogous to the deduction of [7], we have

$$[\cosh(a\mathrm{D}_n) - 1]f_0\cdot f_2 = 0 \tag{29}$$
$$a\mathrm{D}_z f_1\cdot f_2 - 2k\cosh(a\mathrm{D}_n)f_0\cdot f_0 = c_1(t,z)f_1^2 \tag{30}$$

where $c_1(t,z)$ is a suitable function of t and z. Next, we have by using (A11), (28), (A12)–(A14), (A3),

$$\begin{aligned}
\sinh(\tfrac{1}{2}a\mathrm{D}_n)&[a^2\mathrm{D}_t f_1\cdot f_2 - k\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0]\cdot f_1^2 \\
&= a^2\mathrm{D}_t[\sinh(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_2]\cdot[\cosh(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_1] \\
&\quad -k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0]\cdot f_1^2 \\
&= a^2k\mathrm{D}_t[\cosh(\tfrac{1}{2}a\mathrm{D}_n)f_0\cdot f_0]\cdot[\cosh(\tfrac{1}{2}a\mathrm{D}_n)f_1\cdot f_1] \\
&\quad -k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0]\cdot f_1^2 \\
&= 2a^2k\cosh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_t f_0\cdot f_1)\cdot f_0 f_1 \\
&\quad -k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0]\cdot f_1^2 \\
&= k\cosh(\tfrac{1}{2}a\mathrm{D}_n)[(-\mathrm{D}_z + \mathrm{D}_z\cosh(a\mathrm{D}_n))f_0\cdot f_1]\cdot f_0 f_1 \\
&\quad -k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0]\cdot f_1^2 \\
&= -k\cosh(\tfrac{1}{2}a\mathrm{D}_n)(\mathrm{D}_z f_0\cdot f_1)\cdot f_0 f_1 \\
&\quad +k\cosh(\tfrac{1}{2}a\mathrm{D}_n)[\cosh(a\mathrm{D}_n)f_0\cdot f_1]\cdot[\mathrm{D}_z f_0\cdot f_1]
\end{aligned}$$

$$
\begin{aligned}
&+k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n)f_0\cdot f_1]\cdot[\mathrm{D}_z f_0\cdot f_1]\\
&-k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\mathrm{D}_z\sinh(a\mathrm{D}_n)f_0\cdot f_1]\cdot f_0 f_1\\
=&-k\mathrm{D}_z\cosh(\tfrac{1}{2}a\mathrm{D}_n)[\cosh(a\mathrm{D}_n)f_0\cdot f_1]\cdot f_0 f_1\\
&-k\sinh(\tfrac{1}{2}a\mathrm{D}_n)[\sinh(a\mathrm{D}_n)f_0\cdot f_1]\cdot[\mathrm{D}_z f_0\cdot f_1]\\
=&\,0
\end{aligned}
$$

which implies that

$$a^2\mathrm{D}_t f_1\cdot f_2 - k\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0 = c_2(t,z)f_1^2 \tag{31}$$

where $c_2(t,z)$ is a suitable function of t and z. Furthermore, we assume that $f_2(n)$ determined by (28) is chosen such that $c_i(t,z)=0$, $i=1,2$. In this case, we have

$$a\mathrm{D}_z f_1\cdot f_2 - 2k\cosh(a\mathrm{D}_n)f_0\cdot f_0 = 0 \tag{32}$$

$$a^2\mathrm{D}_t f_1\cdot f_2 - k\mathrm{D}_z\exp(a\mathrm{D}_n)f_0\cdot f_0 = 0. \tag{33}$$

By use of (32) and similar to the deduction of [7], we can show

$$[a\mathrm{D}_z - \sinh(a\mathrm{D}_n)]f_0\cdot f_2 = 0. \tag{34}$$

Finally, we have by using (A15) and (A16)

$$
\begin{aligned}
&-\{[2a^3\mathrm{D}_t + a\mathrm{D}_z - a\mathrm{D}_z\cosh(a\mathrm{D}_n)]f_0\cdot f_2\}f_1\\
&\quad=\{[2a^3\mathrm{D}_t + a\mathrm{D}_z - a\mathrm{D}_z\cosh(a\mathrm{D}_n)]f_0\cdot f_1\}f_2\\
&\qquad-\{[2a^3\mathrm{D}_t + a\mathrm{D}_z - a\mathrm{D}_z\cosh(a\mathrm{D}_n)]f_0\cdot f_2\}f_1\\
&\qquad-a\{[\cosh(a\mathrm{D}_n)-1]f_0\cdot f_1\}_z f_2 + a\{[\cosh(a\mathrm{D}_n)-1]f_0\cdot f_2\}_z f_1\\
&\quad=-2a^3 f_0\mathrm{D}_t f_1\cdot f_2 - a\frac{\partial f_0}{\partial z}(n+a)[f_1(n-a)f_2(n)-f_1(n)f_2(n-a)]\\
&\qquad-a\frac{\partial f_0}{\partial z}(n-a)[f_1(n+a)f_2(n)-f_1(n)f_2(n+a)]\\
&\quad=-2a^3 f_0\mathrm{D}_t f_1\cdot f_2 + 2a\frac{\partial f_0}{\partial z}(n+a)\exp\left(-\frac{1}{2}a\frac{\partial}{\partial n}\right)\sinh\left(\frac{1}{2}a\mathrm{D}_n\right)f_1\cdot f_2\\
&\qquad-2a\frac{\partial f_0}{\partial z}(n-a)\exp\left(\frac{1}{2}a\frac{\partial}{\partial n}\right)\sinh\left(\frac{1}{2}a\mathrm{D}_n\right)f_1\cdot f_2\\
&\quad=-2a^3 f_0\mathrm{D}_t f_1\cdot f_2 + 2ak\frac{\partial f_0}{\partial z}(n+a)f_0(n)f_0(n-a)\\
&\qquad-2ak\frac{\partial f_0}{\partial z}(n-a)f_0(n)f_0(n+a)\\
&\quad=0
\end{aligned}
$$

which implies that

$$[2a^3\mathrm{D}_t + a\mathrm{D}_z - a\mathrm{D}_z\cosh(a\mathrm{D}_n)]f_0\cdot f_2 = 0. \tag{35}$$

Thus we have shown the nonlinear superposition formula (28) for equations (5) and (6) under the conditions $c_1(t,z)=c_2(t,z)=0$, and $f_2(n)$ is a new solution.

To summarize, we can seek particular solutions of the differential-difference KdV equations (5) and (6) via the following steps. First, choose a given solution $f_1(n)$ of (5) and (6). Second, from the Bäcklund transformation (25)–(27) we find $f_0(n)$ such that $f_0(n)\to f_1(n)$ and, furthermore, obtain a particular solution $\tilde{f}_2(n)$ from (28). Then a general solution of (28) is $f_2(n)=c(t,z)f_1(n)+\tilde{f}_2(n)$, where $c(t,z)$ is an arbitrary function of t, z. Finally, we substitute $f_2(n)$ into (30) and (31). If $c(t,z)$ can be determined such that $c_1(t,z) = c_2(t,z) = 0$, the corresponding $f_2(n)$ is a new solution of the differential-difference KdV equation (5) and (6). As an application of the obtained result, we can obtain a sequence of polynomial solutions of (5) and (6). For example, if we choose $f_0(n) = n+z$ and $f_1(n) = 1$, then it is easily verified that $n+z$ and 1 are two solutions of (5) and (6) and $n+z \to 1$. Furthermore, we can show that

$$f_2(n) = (n+z)^3 - a^2(n+z) - 2a^2z - 3t + A$$

satisfies (32) and (33) with $c = -\frac{3}{2}a$, where A is an arbitrary constant. Further suppose

$$f_0(n) = (n+z)^3 - a^2(n+z) - 2a^2z - 3t + A \qquad f_1(n) = n+z.$$

Then we seek a solution in the form

$$f_2(n) = (n+z)^6 + a_1(t,z)(n+z)^5 + a_2(t,z)(n+z)^4 + a_3(t,z)(n+z)^3 + a_4(t,z)(n+z)^2 + a_5(t,z)(n+z) + a_6(t,z)$$

such that (28), (32) and (33) hold. A direct calculation shows that

$$k = -\frac{5}{2}a \qquad a_1 = 0 \qquad a_2 = -5a^2 \qquad a_3 = 5[-2a^2z - 3t + A]$$
$$a_4 = 4a^4 \qquad a_5 = 16a^4z + 60a^2t + B$$
$$a_6 = -5A^2 + 30At - 45t^2 + 20Aa^2z - 60a^2tz - 20a^4z^2$$

where A and B are arbitrary constants. In this way, we may deduce a sequence of polynomial solutions of (5) and (6) and so the rational solutions of (1).

5. Conclusion and discussion

In this paper, we have given two different Bäcklund transformations of (5) and (6). Furthermore, corresponding nonlinear superposition formulae have been shown. As a result, N-soliton solutions first obtained by Ohta and Hirota are rederived, and a sequence of rational solutions are also given. Our results provide further evidence that equations (5) and (6) or equivalently (1), are completely integrable. It is noted that if we take (5) as a differential-difference analogue of the KdV equation, (6) is naturally viewed as a higher-order version of (5) in this sense.

Acknowledgments

The work of XBH was supported by the Royal Society, Chinese Academy of Sciences and National Natural Science Foundation of China.

Appendix

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$[\cosh(\delta D_n)b\cdot b] - [D_z\sinh(\delta D_n)a\cdot a] - [\cosh(\delta D_n)a\cdot a][D_z\sinh(\delta D_n)b\cdot b] = 2\sinh(\delta D_n)(D_t a\cdot b)\cdot ab \qquad \text{(A1)}$$

$$[\cosh(3\delta D_n)a\cdot a][\cosh(\delta D_n)b\cdot b] - [\cosh(\delta D_n)a\cdot a][\cosh(3\delta D_n)b\cdot b] = 4\sinh(\delta D_n)[\sinh(2\delta D_n)a\cdot b]\cdot[\cosh(2\delta D_n)a\cdot b] = 2\sinh(2\delta D_n)[\exp(\delta D_n)a\cdot b]\cdot[\exp(-\delta D_n)a\cdot b] \qquad \text{(A2)}$$

$$\sinh(\delta D_n)a\cdot a = 0 \qquad \text{(A3)}$$

$$[\cosh(\delta\,\mathrm{D}_n)b\cdot b][\mathrm{D}_z\sinh(3\delta\,\mathrm{D}_n)a\cdot a]-[\cosh(\delta\,\mathrm{D}_n)a\cdot a][\mathrm{D}_z\sinh(3\delta\,\mathrm{D}_n)b\cdot b]$$
$$=2\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\cosh(2\delta\,\mathrm{D}_n)a\cdot b]$$
$$+2\sinh(\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)a\cdot b][\cosh(2\delta\,\mathrm{D}_n)a\cdot b]$$
$$+[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\}$$
$$=\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)[\exp(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(-\delta\,\mathrm{D}_n)a\cdot b]$$
$$+\sinh(2\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\exp(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(-\delta\,\mathrm{D}_n)a\cdot b]$$
$$-[\exp(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z\exp(-\delta\,\mathrm{D}_n)a\cdot b]\} \quad \text{(A4)}$$

$$[\cosh(3\delta\,\mathrm{D}_n)b\cdot b][\mathrm{D}_z\sinh(\delta\,\mathrm{D}_n)a\cdot a]-[\cosh(3\delta\,\mathrm{D}_n)a\cdot a][\mathrm{D}_z\sinh(\delta\,\mathrm{D}_n)b\cdot b]$$
$$=2\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\cosh(2\delta\,\mathrm{D}_n)a\cdot b]$$
$$-2\sinh(\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)a\cdot b][\cosh(2\delta\,\mathrm{D}_n)a\cdot b]$$
$$+[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\}$$
$$=\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)[\exp(-\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(\delta\,\mathrm{D}_n)a\cdot b]$$
$$+\sinh(2\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\exp(-\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(\delta\,\mathrm{D}_n)a\cdot b]$$
$$-[\exp(-\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z\exp(\delta\,\mathrm{D}_n)a\cdot b]\} \quad \text{(A5)}$$

$$\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab$$
$$=\sinh(\delta\,\mathrm{D}_n)\{(\mathrm{D}_z a\cdot b)\cdot[\cosh(2\delta\,\mathrm{D}_n)a\cdot b]+[\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab\} \quad \text{(A6)}$$

$$2\sinh(\delta\,\mathrm{D}_n)(\mathrm{D}_t a\cdot b)\cdot ab=\mathrm{D}_t[\exp(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(-\delta\,\mathrm{D}_n)a\cdot b] \quad \text{(A7)}$$

$$\mathrm{D}_t[\exp(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(-\delta\,\mathrm{D}_n)c\cdot d]=\exp(\delta\,\mathrm{D}_n)[(\mathrm{D}_t a\cdot d)\cdot cb-ad\cdot(\mathrm{D}_t c\cdot b)] \quad \text{(A8)}$$

$$2\sinh(2\delta\,\mathrm{D}_n)[\exp(3\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(\delta\,\mathrm{D}_n)c\cdot d]$$
$$=\exp(\delta\,\mathrm{D}_n)\{[\exp(4\delta\,\mathrm{D}_n)a\cdot d]\cdot cb-ad\cdot[\exp(4\delta\,\mathrm{D}_n)c\cdot b]\} \quad \text{(A9)}$$

$$2\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)[\exp(3\delta\,\mathrm{D}_n)a\cdot b]\cdot[\exp(\delta\,\mathrm{D}_n)c\cdot d]$$
$$=\exp(\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\exp(4\delta\,\mathrm{D}_n)a\cdot d]\cdot cb-[\exp(4\delta\,\mathrm{D}_n)a\cdot d]\cdot(\mathrm{D}_z c\cdot b)\}$$
$$+\exp(\delta\,\mathrm{D}_n)\{(\mathrm{D}_z a\cdot d)\cdot[\exp(4\delta\,\mathrm{D}_n)c\cdot b]-ad\cdot[\mathrm{D}_z\exp(4\delta\,\mathrm{D}_n)c\cdot b]\} \quad \text{(A10)}$$

$$\sinh(\delta\,\mathrm{D}_n)(\mathrm{D}_t a\cdot b)\cdot a^2=\mathrm{D}_t[\sinh(\delta\,\mathrm{D}_n)a\cdot b]\cdot[\cosh(\delta\,\mathrm{D}_n)a\cdot a] \quad \text{(A11)}$$

$$\mathrm{D}_t[\cosh(\delta\,\mathrm{D}_n)a\cdot a]\cdot[\cosh(\delta\,\mathrm{D}_n)b\cdot b]=2\cosh(\delta\,\mathrm{D}_n)(\mathrm{D}_t a\cdot b)\cdot ab \quad \text{(A12)}$$

$$\sinh(\delta\,\mathrm{D}_n)[\mathrm{D}_z\exp(2\delta\,\mathrm{D}_n)a\cdot a]\cdot b^2$$
$$=\cosh(\delta\,\mathrm{D}_n)[\mathrm{D}_z\cosh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab$$
$$+\sinh(\delta\,\mathrm{D}_n)[\mathrm{D}_z\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab$$
$$-\cosh(\delta\,\mathrm{D}_n)[\cosh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z a\cdot b]$$
$$-\sinh(\delta\,\mathrm{D}_n)[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z a\cdot b] \quad \text{(A13)}$$

$$\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)[\cosh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab$$
$$=\sinh(\delta\,\mathrm{D}_n)\{[\mathrm{D}_z\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot ab-[\sinh(2\delta\,\mathrm{D}_n)a\cdot b]\cdot[\mathrm{D}_z a\cdot b]\} \quad \text{(A14)}$$

$$(\mathrm{D}_t a\cdot b)c-(\mathrm{D}_t a\cdot c)b=-a\mathrm{D}_t b\cdot c \quad \text{(A15)}$$

$$[\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)a\cdot b]c-[\mathrm{D}_z\cosh(\delta\,\mathrm{D}_n)a\cdot c]b+\frac{\partial}{\partial z}[\cosh(\delta\,\mathrm{D}_n)a\cdot b]c$$
$$-\frac{\partial}{\partial z}[\cosh(\delta\,\mathrm{D}_n)a\cdot c]b=\frac{\partial a}{\partial z}(n+\delta)[b(n-\delta)c(n)-b(n)c(n-\delta)]$$
$$+\frac{\partial a}{\partial z}(n-\delta)[b(n+\delta)c(n)-b(n)c(n+\delta)]. \quad \text{(A16)}$$

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